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Symmetry-preserving difference schemes for some heat transfer equations

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Abstract. Lie group analysis of differential equations is a generally recognized method, which provides invariant solutions, integrability, conservation laws etc. In this paper we present three characteristic examples of the construction of invariant difference equations and meshes, where the original continuous symmetries are preserved in discrete models. Conservation of symmetries in difference modelling helps to retain qualitative properties of the differential equations in their difference counterparts.

1. Introduction

Symmetries are intrinsic and fundamental features of the differential equations of mathematical physics. Consequently, they should be retained when discrete analogues of such equations are constructed.

The group properties of a heat-transfer equation with a source

$$u_t = (K(u)u_x)_x + Q(u) \tag{1}$$

were considered in [4], and all choices of $K(u)$ and $Q(u)$ which extend the symmetry group admitted by the general case of equation (1) were identified. In this paper we consider two partial cases of nonlinearities from the classification in [4]:

$$u_t = (u^\sigma u_x)_x \pm u^n \qquad \sigma, n = \text{constant} \tag{2}$$

$$u_t = u_{xx} + \delta u \ln u \qquad \delta = \pm 1 \tag{3}$$

together with linear case

$$u_t = u_{xx} \tag{4}$$

whose group properties were known by Lie. For all cases we construct difference equations and meshes (lattices) that admit the same Lie group of point transformations as their continuous limits.

We recall that Lie point symmetries yield a number of useful properties of differential equations [13, 16, 10].

(a) A group action transforms the complete set of solutions into itself; so it is possible to obtain new solutions from a given one.

(b) There exists a standard procedure to obtain the whole set of invariants and differential invariants for a symmetry group; it yields the invariant representation of the differential

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equations and the forms of invariant solutions in which they could be found (symmetry reduction of partial differential equations (PDEs).

(c) For ordinary differential equations (ODEs) the known symmetry yields the reduction of the order; if the dimension of symmetry is equal to (or greater than) the order of the ODE, then we have complete integrability.

(d) The invariance of PDEs is a necessary condition for the application of Noether's theorem on variational problems to obtain conservation laws.

(e) It should be noted that Lie point transformations have a clear geometrical interpretation and one can construct the orbits of a group in a finite-dimensional space of independent and dependent variables.

The structure of the admitted group essentially affects the construction of equations and grids. Group transformations can break the geometric structure of the mesh that influences the approximation and other properties of a difference equation. Early contributions to the construction of the difference grids based on the symmetries of the initial difference model are [6, 8]. Classes of transformations that conserve uniformity, orthogonality, and other properties of meshes will be defined below.

In accordance with equation (1) we consider Lie point transformations in a space with two independent variables: t and x. Let

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \cdots \tag{5}$$

be an operator of a one-parameter transformation group. Dots denote prolongation of the operator on other variables used in the given differential equation:

$$F(x, t, u_t, u, u_x, u_{xx}) = 0. \tag{6}$$

The group generated by (5) transforms a point $(x, t, u, u_t, u_x, u_{xx})$ to a new one $(x^*, t^*, u^*, u_t^*, u_x^*, u_{xx}^*)$ together with equation (6). This situation changes when applying Lie point, transformations to difference equations. Let

$$F(z) = 0 \tag{7}$$

be a difference equation defined on some finite set of points $z^1, z^2, \ldots$ (difference stencil) on a mesh. In contrast to the point $(x, t, u, u_t, u_x, u_{xx})$, the 'difference point'—the difference stencil has its own geometrical structure.

Let

$$\Omega(z, h) = 0 \tag{8}$$

be an equation that defines a difference stencil and a mesh. As an example it will be a uniform mesh if the left step (spacing) equals the right step:

$$h^+ = h^-. \tag{9}$$

The invariance of the difference equation (7) depends on the invariance of (8), since the latter must be included in the general condition of invariance:

$$\begin{cases} XF(z)_{|(7)(8)} & = 0 \\ X\Omega(z, h)_{|(7)(8)} & = 0. \end{cases} \tag{10}$$

Relations between the two conditions in (10) depend on whether Ω, ξ^t and ξ^x depend on the solution or not. If $\Omega_u = \xi_u^t = \xi_u^x = 0$, then the conditions (10) could be considered independently.

Thus, what makes our approach [5–8] special is the inclusion of the second equation of (10) in the conditions of invariance, admitting the whole set of properties (a)–(e) stated for equations (7) and (8).

There exist a few ways to avoid transformations of the difference stencil and, consequently, transformations of a mesh. One way is to restrict transformations to the case when independent variables are not changed: $\xi^t = \xi^x = 0$, yielding any mesh invariant (see [12]). But this restriction is very strong and would exclude most symmetries of physical problems.

Another approach is connected to evolutionary vector fields. It is known [10, 16] that the symmetry operator could be represented as

$$\bar{X} = (\eta - \xi^t u_t - \xi^x u_x)\frac{\partial}{\partial u} + \cdots \tag{11}$$

which could be viewed as representative of a factor algebra by the ideal

$$\xi^t(z)D_t + \xi^x(z)D_x \tag{12}$$

where $D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + \cdots$ and $D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u}\cdots$ are both admitted by all equations (6). For differential equations this approach gives an equivalent result but with some losses in points (a)–(e). In particular we lose the geometric sense even for point transformations due to the realization of group transformations in infinite-dimensional spaces $(t, x, u_t, u_x, u_{tx}, u_{xx}, u_{tt}, \cdots)$, there is no procedure for calculating invariants of operators of the type given by (11) etc.

As was shown in [5], the exact representation of the operator of total differentiation D_x in a space of difference variables is given by the following operators (the same is also true for D_t):

$$\begin{aligned} D^+ &= \frac{\partial}{\partial x} + \underset{+h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \cdots \\ D^- &= \frac{\partial}{\partial x} + \underset{-h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \cdots \end{aligned} \tag{13}$$

where $\underset{+h}{\tilde{D}} = \sum_{n=1}^{\infty}\frac{(-h)^{n-1}}{n}\underset{+h}{D}^n$, $\underset{-h}{\tilde{D}} = \sum_{n=1}^{\infty}\frac{h^{n-1}}{n}\underset{-h}{D}^n$ and $\underset{+h}{D}$, $\underset{-h}{D}$ are right and left difference operators on a uniform mesh. As difference operators are related to corresponding shift operators $\underset{+h}{S}$, $\underset{-h}{S}$:

$$\underset{\pm h}{D} = \pm\frac{(\underset{\pm h}{S} - 1)}{h}$$

we obtain another representation for $\underset{\pm h}{\tilde{D}}$:

$$\underset{+h}{\tilde{D}} = \sum_{n=1}^{\infty}\frac{(-h)^{n-1}}{n}\underset{+h}{D}^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{nh}(\underset{+h}{S} - 1)^n \tag{14}$$

$$\underset{-h}{\tilde{D}} = \sum_{n=1}^{\infty}\frac{(1 - \underset{-h}{S})^n}{nh}. \tag{15}$$

The group transformation for operators (13) can be obtained by the exponential mapping or by means of the so-called Newton series (see [5]). It is important to note that every difference equation on a regular mesh admits the operators D^+ and D^-, which do not change a mesh. It was shown [5], that a family of operators $\xi(z)D^{\pm}$ forms an ideal in the Lie algebra of operators (5), since it is possible to rewrite these operators as evolutionary vector fields

$$\bar{X} = (\eta - \xi^t D_t^+(u) - \xi^x D_x^+(u))\frac{\partial}{\partial u} + \cdots. \tag{16}$$

(We follow here the right semiaxis representation D^+. The same formulation can be done with help of left semiaxis representation D^-.)

It is important to notice that the representation (13) is true only for regular (uniform) meshes. Thus, the evolutionary vector fields (16) are applicable only for the groups which do not change the uniformity of a mesh. So, one cannot apply them for modern numerical methods with moving meshes, self adaptive meshes or multigrid methods etc.

Let us consider a one-parameter transformation group which is generated by the operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \cdots. \tag{17}$$

We prolong (17) on the right and left steps h^+, h^- by means of relations $h^+ = x^+ - x$ and $h^- = x - x^-$, where $f^\pm \equiv \underset{\pm h}{S}(f)$:

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \cdots + [\underset{+h}{S}(\xi) - \xi]\frac{\partial}{\partial h^+} + [\xi - \underset{-h}{S}(\xi)]\frac{\partial}{\partial h^-}. \tag{18}$$

From (18) it is easy to obtain the invariance condition for a uniform mesh in a given direction. Let (9) be invariant with respect to (18), then

$$\underset{+h}{S}(\xi) - 2\xi + \underset{-h}{S}(\xi) = 0 \qquad \text{or} \qquad \underset{+h}{D}\,\underset{-h}{D}(\xi) = 0. \tag{19}$$

Condition (19) is a strong limitation on the admitted group. In addition the coefficients of (16) are the power series of $\underset{\pm h}{D}$ or $\underset{\pm h}{S}$, since one should consider the whole set of mesh points and not a stencil only.

Let us illustrate the above with a simple example. The ODE

$$u_{xx} = u^2 \tag{20}$$

can be viewed as the stationary case of equation (2) with $\sigma = 0$, $n = 2$. Equation (20) has the following Lie point symmetries:

$$X_1 = \frac{\partial}{\partial x} \qquad X_2 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{21}$$

As a difference analogue of (20) we consider

$$\frac{u^+ - 2u + u^-}{h^2} = u^2 \tag{22}$$

on a uniform mesh

$$h^+ = h^- \tag{23}$$

where $u^+ = \underset{+h}{S}(u)$ and $u^- = \underset{-h}{S}(u)$.

Equations (22) and (23) use a three-point stencil or subspace $(x, x^+, x^-, u, u^+, u^-)$ and the operators (21) have the following prolongation for the shifted points of the difference stencil:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-} \\ X_2 &= x\frac{\partial}{\partial x} + x^+\frac{\partial}{\partial x^+} + x^-\frac{\partial}{\partial x^-} - 2u\frac{\partial}{\partial u} - 2u^+\frac{\partial}{\partial u^+} - 2u^-\frac{\partial}{\partial u^-} + h^+\frac{\partial}{\partial h^+} + h^-\frac{\partial}{\partial h^-}. \end{aligned} \tag{24}$$

The symmetry algebra (21) acts on the space (t, x), so the coefficients of (24) have the same form in different points of the stencil. The prolongation forms for h^+ and h^- are easily derived from the relations $h^+ = x^+ - x$, $h^- = x - x^-$.

It is easy to verify that equations (22) and (23) are invariant under the action of (24):

$$
\begin{aligned}
&X_2\left(\frac{u^+ - 2u + u^-}{h^2} - u^2\right)_{|(22)} = 0 \\
&X_2(h^+ - h^-)_{|(23)} = 0.
\end{aligned}
\tag{25}
$$

(Operator X_1 leaves (22) and (23) unchanged.)

It follows that the system (22) and (23) has the same Lie point symmetry as its continuous limit. Note that the invariance conditions (25) are mutually independent.

As for the continuous case one can easily calculate the finite-difference invariants for (24) by solving the standard system:

$$X_i I^\tau(x, h^+, h^-, u, u^+, u^-) = 0 \qquad i = 1, 2 \qquad \tau = 1, 2, 3, 4. \tag{26}$$

The solution of (26) yields the whole set of difference invariants

$$I^1 = \frac{h^+}{h^-} \qquad I^2 = \frac{u^+}{u} \qquad I^3 = \frac{u^-}{u} \qquad I^4 = (h^+)^2 u. \tag{27}$$

It follows that the difference model (22) and (23) can be represented by means of the invariants (27) as $I^2 + I^3 - 2 = I^4$ and $I^1 = 1$.

Let us now consider the evolutionary vector fields for the difference equation (22) (we consider the right semiaxis representation D^+). We prolong the operator (13) on all points of a given stencil $(x, x^+, x^-, u, u^+, u^-)$:

$$D^+ = \frac{\partial}{\partial x} + \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-} + u_x \frac{\partial}{\partial u} + u_x^+ \frac{\partial}{\partial u^+} + u_x^- \frac{\partial}{\partial u^-} \tag{28}$$

where

$$u_x \equiv \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{D^n}(u) \qquad u_x^+ \equiv \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{D^n}(u^+)$$

and

$$u_x^- \equiv \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{D^n}(u^-).$$

The evolution vector fields (16) will have the following forms:

$$
\begin{aligned}
\underset{h}{\bar{X}_1} &= -X_1 + D^+ = u_x \frac{\partial}{\partial u} + u_x^+ \frac{\partial}{\partial u^+} + u_x^- \frac{\partial}{\partial u^-} \\
\underset{h}{\bar{X}_2} &= -X_2 + xD^+ = -h^+ \frac{\partial}{\partial h^+} - h^- \frac{\partial}{\partial h^-} + (2u + xu_x)\frac{\partial}{\partial u} + (2u^+ + xu_x^+)\frac{\partial}{\partial u^+} \\
&\quad + (2u^- + xu_x^-)\frac{\partial}{\partial u^-}.
\end{aligned}
\tag{29}
$$

It is not easy to check the invariance conditions of equation (22) for the operators (29) because one should use not only equation (22), but all its sequences obtained by shifting to the right. A harder question is how to produce the finite-difference invariants (27) by means of (29). That is why we prefer the first classical method for Lie point symmetries and include a mesh in the invariance condition (10) (a very similar approach for the semi-discretized nonlinear heat equation was introduced recently in [3]). An additional convincing reason to apply the classical representation of Lie point symmetries springs from comparison of unsuitable evolutionary vector fields approaches with invariant variational problems, developed in [5], and a clear classical way to construct Noether type theorems for difference equations [7].

Another approach to the symmetry of difference equations on a fixed uniform mesh was introduced in [9]. However, that method is only applicable to linear equations; moreover, it requires knowing a complete set of solutions of difference equations. The newly introduced approach [11] deals with evolutionary vector fields on a uniform mesh. The advantage of the last two approaches seems to be in the potential of finding non-point symmetries of difference equations which are not available in their continuous limits.

Returning to equation (1), we note that a transformation defined by (5) conserves uniformity of a grid in t and x directions, if

$$\underset{+\tau}{D}\,\underset{-\tau}{D}(\xi^t) = 0 \tag{30}$$

$$\underset{+h}{D}\,\underset{-h}{D}(\xi^x) = 0 \tag{31}$$

where $\underset{+\tau}{D}$ and $\underset{-\tau}{D}$ are right and left difference operators in the t direction.

Conditions (30) and (31) are not sufficient to describe the invariance of an orthogonal mesh. For an orthogonal mesh to conserve its orthogonality under the transformation, it is necessary and sufficient that [6, 8]:

$$\underset{+h}{D}(\xi^t) = -\underset{+\tau}{D}(\xi^x). \tag{32}$$

When condition (32) is not satisfied for a given group, the flatness of the layer of a grid in some direction is rather important. For evolution equations it is significant to have flat time layers, since otherwise, after transformations, some domains of a space could be in the future, while others in the past. We have a simple criterion [6, 8] for preserving the flatness of the layer of a grid in the time direction under the action of a given operator (5):

$$\underset{+h}{D}\,\underset{+\tau}{D}(\xi^t) = 0. \tag{33}$$

So, the conditions (30)–(33) provide a geometry of grids that is based on the Lie group symmetry. These conditions will be used in what follows.

2. Invariant model for the equation $u_t = (u^\sigma u_x)_x \pm u^n$

The equation

$$u_t = (u^\sigma u_x)_x \pm u^n \qquad \sigma, n = \text{constant} \tag{34}$$

admits a 3-parameter symmetry group. This group can be represented by the following infinitesimal operators [4]:

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial x}\, X_3 = 2(n-1)t\frac{\partial}{\partial t} + (n-\sigma-1)x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{35}$$

The set (35) satisfied all conditions (30)–(33). So, we can use an orthogonal grid that is uniform in the t and x directions. Let us consider the set of operators (35) in the space $(t, \hat{t}, x, h^+, h^-, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$ that corresponds to the stencil shown in figure 1.

There are seven difference invariants of the Lie algebra (35):

$$\frac{\tau^{\frac{n-\sigma-1}{2(n-1)}}}{h} \qquad \tau u^{n-1} \qquad \frac{\hat{u}}{u} \qquad \frac{u_+}{u} \qquad \frac{u_-}{u} \qquad \frac{\hat{u}_+}{\hat{u}} \qquad \frac{\hat{u}_-}{\hat{u}}. \tag{36}$$

The small number of symmetry operators (35) provides us with a large number of difference invariants (36). Thus we are left with some additional degrees of freedom in invariant

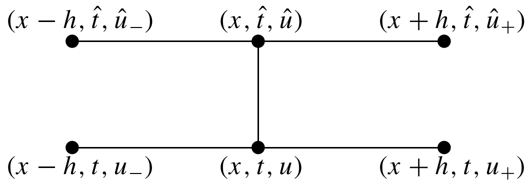

Figure 1.

difference modelling of (34). By means of the invariants (36), we could write the following explicit scheme for (34):

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}\left(\left(\frac{u_+ + u}{2}\right)^{\sigma} \underset{h}{u}_x - \left(\frac{u+u_-}{2}\right)^{\sigma} \underset{h}{u}_{\bar{x}}\right) \pm u^n \tag{37}$$

where $\underset{h}{u}_x = \frac{u_+ - u}{h}$, $\underset{h}{u}_{\bar{x}} = \frac{u-u_-}{h}$.

This scheme is not unique and one could construct another form of invariant difference equation. For example an implicit scheme could be as follows:

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h^2}(\hat{u}_+^{\sigma+1} - 2\hat{u}^{\sigma+1} + \hat{u}_-^{\sigma+1}) + \hat{u}^n. \tag{38}$$

Note, that the continuous limit of the last difference equation

$$u_t = (u^{\sigma+1})_{xx} + u^n. \tag{39}$$

is equivalent to equation (34) up to the scaling of x. But scheme (38) is not equivalent to scheme (37), because there is no point transformation that relates them. In [18] Samarskii *et al* considered the case $n = \sigma + 1$ and found a finite-difference blow-up solution for equation (38) that is invariant with respect to the operator

$$X_3^* = (t - T_0)\frac{\partial}{\partial t} - \frac{1}{\sigma}u\frac{\partial}{\partial u} \tag{40}$$

where T_0 is constant. The operator (40) defines a subgroup which is equivalent to the self-similar subgroup with the operator X_3 of the set (35). Let us find the solution of the problem (38) in the invariant form:

$$u = \left(1 - \frac{t}{T_0}\right)^{-1/\sigma} \theta(x).$$

This solution is sought [18] on the set of infinite number of time intervals on $[0, T_0]$:

$$\tau_j = T_0 \frac{\rho^{\sigma} + 1}{\rho^{\sigma}} \rho^{-\sigma j} \tag{41}$$

where $\rho > 1$ is constant.

For the function $\theta(x)$ we have the equation

$$(\theta^{\sigma+1})_{\bar{x}x} + \theta^{\sigma+1} = \frac{1}{\sigma}\theta. \tag{42}$$

The solution of the problem (42) was found in [18] for the case $\sigma = 2$. Let us fix an arbitrary $M > 0$ and let $h = 2\sin(\frac{3\pi}{2(M+1)})$. In this case the localization length equals

$$l_h = \frac{3\pi h}{2}\left(\arcsin\frac{h}{2}\right)^{-1} \qquad 0 < h \leqslant 2$$

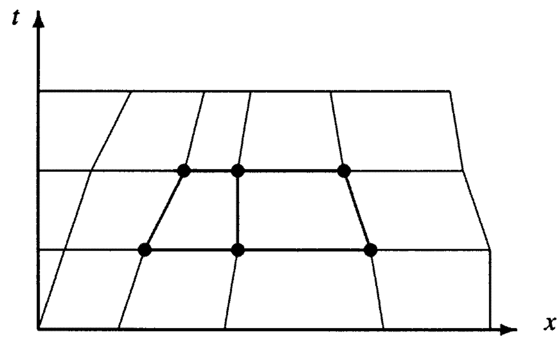

Figure 2.

(see [18]). Then, one can verify that the solution of the problem (42) in the points $x_k = kh$ has the form

$$\theta_{kh} = \sqrt{2}\left(3\left(1 - \frac{4}{h^2}\sin^2\frac{a_h h}{2}\right)\right)^{-1/2}\sin(a_h kh) \qquad k = 0, 1, \ldots, M+1 \tag{43}$$

where $a_h = \pi/l_h$.

The obtained function u gives the blow-up solution of the problem in the case $\sigma = 2$, $\beta = 3$, $l = l_h$. This solution tends to infinity in all points of the space grid, conserving the structure. As $h \to 0$, the difference solution (43) tends to the solution of the ODE:

$$\theta(x) = \left(\frac{3}{4}\right)^{1/2}\sin\left(\frac{x}{3}\right) \qquad 0 < x < l_0 = 3\pi.$$

3. Invariant difference model for the semilinear heat-transfer equation

The semilinear heat-transfer equation

$$u_t = u_{xx} + \delta u \ln u \qquad \delta = \pm 1 \tag{44}$$

admits the 4-parameter Lie symmetry group of point transformations [4], that corresponds to the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial x} \qquad X_3 = 2\mathrm{e}^{\delta t}\frac{\partial}{\partial x} - \delta \mathrm{e}^{\delta t} x u \frac{\partial}{\partial u} \qquad X_4 = \mathrm{e}^{\delta t} u \frac{\partial}{\partial u}. \tag{45}$$

Before constructing a difference equation and grid that approximate (44) and inherit the whole Lie algebra (45), we should first check condition (32) for the invariance of orthogonality. The operators X_1, X_2 and X_4 conserve orthogonality, but X_3 does not: condition (32) is not true for the last operator. Consequently an orthogonal mesh cannot be used for the invariant modelling of (44).

The conditions (33) are true for the complete set (45), so it is possible to use a nonorthogonal grid with flat time layers. An example of a grid with with flat time layer is shown in figure 2.

A possible reformulation of equation (44) by using the four differential invariants in the subspace $(t, x, u, u_x, u_{xx}, \mathrm{d}t, \mathrm{d}x, \mathrm{d}u)$:

$$J^1 = \mathrm{d}t \qquad J^2 = \left(\frac{u_x}{u}\right)^2 - \frac{u_{xx}}{u} \qquad J^3 = 2\frac{u_x}{u} + \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$J^4 = \frac{\mathrm{d}u}{u\mathrm{d}t} - \delta \ln u + \frac{1}{4}(\mathrm{d}x/\mathrm{d}t)^2.$$

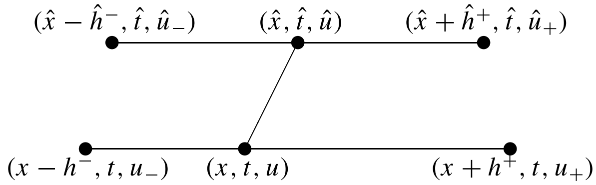

Figure 3.

is given by the system:

$$\begin{cases} J^3 = 0 \\ J^4 = J^2 \end{cases}$$

or

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2\dfrac{u_x}{u} \\ \dfrac{\mathrm{d}u}{\mathrm{d}t} = u_{xx} + \delta u \ln u - 2\dfrac{u_x^2}{u}. \end{cases} \tag{46}$$

So, the structure of the group (45) suggests the use of two evolution equations.

As the next step, we will find difference invariants for the set X_1–X_4 of the group (45). These invariants are necessary for the approximation of the system (46). We will use the 6-point difference stencil, as shown in figure 3 on which we will approximate the system (46). The stencil defines the difference subspace $(t, \hat{t}, x, \hat{x}, h^+, h^-, \hat{h}^+, \hat{h}^-, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$. The group (45) has the following difference invariants in this subspace:

$$I^1 = \tau \qquad I^2 = h^+ \qquad I^3 = h^- \qquad I^4 = \hat{h}^+ \qquad I^5 = \hat{h}^-$$

$$I^6 = (\ln u)_x - (\ln u)_{\bar{x}} \qquad I^7 = (\ln \hat{u})_x - (\ln \hat{u})_{\bar{x}}$$

$$I^8 = \delta \Delta x + 2(\mathrm{e}^{\delta\tau} - 1)\left(\frac{h^-}{h^+ + h^-}(\ln u)_x + \frac{h^+}{h^+ + h^-}(\ln u)_{\bar{x}}\right)$$

$$I^9 = \delta \Delta x + 2(1 - \mathrm{e}^{-\delta\tau})\left(\frac{\hat{h}^-}{\hat{h}^+ + \hat{h}^-}(\ln \hat{u})_x + \frac{\hat{h}^+}{\hat{h}^+ + \hat{h}^-}(\ln \hat{u})_{\bar{x}}\right)$$

$$I^{10} = \delta(\Delta x)^2 + 4(1 - \mathrm{e}^{-\delta\tau})(\ln \hat{u} - \mathrm{e}^{\delta\tau} \ln u)$$

where $\Delta x = \hat{x} - x$, $(\ln u)_x = \frac{\ln u_+ - \ln u}{h^+}$ and $(\ln u)_{\bar{x}} = \frac{\ln u - \ln u_-}{h^-}$.

To obtain an invariant difference model, it is natural to use the difference invariants. An explicit model is given by

$$\begin{cases} I^8 = 0 \\ I^{10} = \dfrac{8}{\delta}\dfrac{(\mathrm{e}^{\delta I^1} - 1)^2}{I^2 + I^3} I^6 \end{cases}$$

or

$$\begin{cases} \delta \Delta x + 2(\mathrm{e}^{\delta\tau} - 1)\left(\dfrac{h^-}{h^+ + h^-}(\ln u)_x + \dfrac{h^+}{h^+ + h^-}(\ln u)_{\bar{x}}\right) = 0 \\ \delta(\Delta x)^2 + 4(1 - \mathrm{e}^{-\delta\tau})(\ln \hat{u} - \mathrm{e}^{\delta\tau} \ln u) = \dfrac{8}{\delta}\dfrac{(\mathrm{e}^{\delta\tau} - 1)^2}{h^+ + h^-}[(\ln u)_x - (\ln u)_{\bar{x}}]. \end{cases} \tag{47}$$

As in the continuous case, there is a reduction in the difference case. When we consider an invariant solution, we have the reduction of the equation-grid system to a system of ODEs.

One being the difference model for the considered equation, the other for the evolution of the grid.

Let us find the solution of the difference model (47) which is invariant with respect to the operator

$$2\alpha X_2 + X_3 \qquad \alpha = \text{constant} \tag{48}$$

$u \exp\left(\frac{\delta e^{\delta t}}{\alpha + e^{\delta t}} \frac{x^2}{4}\right)$, $\left(\frac{\Delta x}{e^{\delta t}(e^{\delta\tau}-1)} - \frac{x}{\alpha + e^{\delta t}}\right)$ and t are all the invariants with respect to (48). Therefore we will seek an invariant solution in the form:

$$\begin{cases} u(x,t) = \exp\left(-\dfrac{\delta e^{\delta t}}{\alpha + e^{\delta t}} \dfrac{x^2}{4}\right) e^{f(t)} \\ \dfrac{\Delta x}{e^{\delta t}(e^{\delta\tau}-1)} = \dfrac{x}{\alpha + e^{\delta t}} + g(t). \end{cases}$$

Substituting this form of the solution into (47), we obtain a system of ODEs to determine $f(t)$ and $g(t)$:

$$\begin{cases} \dfrac{f(t+\tau) - e^{\delta\tau} f(t)}{e}^{\delta\tau}(e^{\delta\tau} - 1) = -\dfrac{1}{2} \dfrac{e^{\delta t}}{\alpha + e^{\delta t}} \\ g(t) = 0. \end{cases}$$

The solution of this system yields the solution of the difference equation (47):

$$u(x,t) = \exp\left(e^{\delta t}\left(f(0) - \frac{e^{\delta\tau} - 1}{2} \sum_{j=1}^{n-1} \frac{e^{-\delta t_j}}{1 + \alpha e^{-\delta t_j}}\right) - \frac{\delta e^{\delta t}}{\alpha + e^{\delta t}} \frac{x^2}{4}\right)$$

and the grid

$$x = x^0 \frac{e^{\delta t} + \alpha}{1 + \alpha}.$$

Here $x = x_i^j = x_i(t_j)$ and $t = t_j$. For $t = 0$ the grid can be arbitrary, but if a regular grid is used it will be regular on every time layer.

The obtained solution is the solution of the Cauchy problem with initial conditions:

$$u(x,0) = \exp\left(f(0) - \frac{\delta e^{\delta t}}{\alpha + e^{\delta t}} \frac{x^2}{4}\right).$$

4. Invariant discrete version of the linear heat equation

The linear heat-transfer equation

$$u_t = u_{xx} \tag{49}$$

admits a 6-parameter Lie symmetry group of point transformations, corresponding to the following infinitesimal operators:

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t} \qquad X_2 = \frac{\partial}{\partial x} \qquad X_3 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u} \qquad X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \\ &X_5 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 + 2t)u\frac{\partial}{\partial u} \qquad X_6 = u\frac{\partial}{\partial u} \end{aligned} \tag{50}$$

and an infinite-dimensional symmetry obtained from the linearity of equation (49):

$$X^* = a(x,t)\frac{\partial}{\partial u}$$

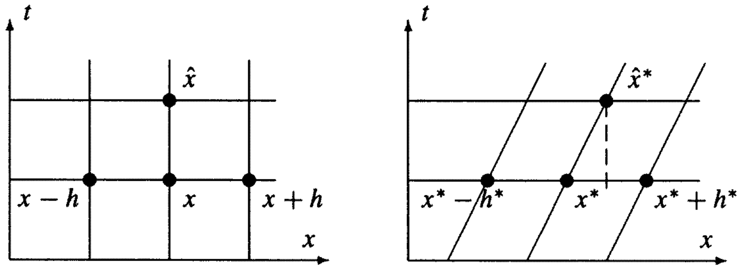

Figure 4.

where $a(t, x)$ in an arbitrary solution of equation (49).

Now we are in a position to show that the invariant difference model for the linear heat-transfer equation cannot be constructed on an orthogonal grid. The model

$$\frac{\hat{u}-u}{\tau} = \frac{u_+ - 2u + u_-}{h^2} \tag{51}$$

on the regular orthogonal mesh, which is used as an invariant model in [2], (see also [1, p 363]), actually does not admit operators X_3 and X_5.

Let us check, for example, the symmetry that is described by X_3. The operator X_3 generates the following transformations

$$\begin{aligned} x^* &= x + 2t\alpha \\ t^* &= t \\ u^* &= u\mathrm{e}^{-x\alpha - t\alpha^2}. \end{aligned} \tag{52}$$

This transformation destroys the orthogonality of the mesh as shown in figure 4.

The transformation (52) transforms the finite-difference equation (51) into the following:

$$\frac{\hat{u}\mathrm{e}^{-\tau\alpha^2} - u}{\tau} = \frac{u_+\mathrm{e}^{-h\alpha} - 2u + u_-\mathrm{e}^{h\alpha}}{h^2} \tag{53}$$

which explicitly depends on a group parameter α. The first differential approximation of equation (53)

$$u_t = u_{xx} - 4\alpha u_x + 2u\alpha^2 + \mathrm{O}(\tau + h)$$

shows explicitly the absence of invariance for equation (51) on an orthogonal mesh.

Consequently, we have to construct a difference model for (49) on a moving mesh.

With help of the differential invariants in the space $(t, x, u, u_x, u_{xx}, \mathrm{d}t, \mathrm{d}x, \mathrm{d}u)$:

$$J^1 = \frac{\mathrm{d}x + 2\frac{u_x}{u}\,\mathrm{d}t}{\mathrm{d}t^{1/2}} \qquad J^2 = \frac{\mathrm{d}u}{u} + \frac{1}{4}\frac{\mathrm{d}x^2}{\mathrm{d}t} + \left(-\frac{u_{xx}}{u} + \frac{u_x^2}{u^2}\right)\mathrm{d}t$$

we can represent the heat equation (49) as the system:

$$\begin{cases} J^1 = 0 \\ J^2 = 0 \end{cases}$$

or

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -2\dfrac{u_x}{u} \\[2mm] \dfrac{\mathrm{d}u}{\mathrm{d}t} = u_{xx} - 2\dfrac{u_x^2}{u}. \end{cases} \tag{54}$$

System (54) inherits the set of operators X_1–X_6, X^*. Implying that it entirely inherits the Lie symmetry group admitted by the linear heat equation (49).

For the difference modelling of the system (54) we need the whole set of difference invariants of the Lie symmetry group (50) in the difference space, corresponding to the chosen stencil $(t, \hat{t}, x, \hat{x}, h^+, h^-, \hat{h}^+, \hat{h}^-, u, \hat{u}, u_+, u_-, \hat{u}_+, \hat{u}_-)$:

$$I^1 = \frac{h^+}{h^-} \qquad I^2 = \frac{\hat{h}^+}{\hat{h}^-} \qquad I^3 = \frac{\hat{h}^+ h^+}{\tau}$$

$$I^4 = \frac{\tau^{1/2}}{h^+}\frac{\hat{u}}{u}\exp\left(\frac{1}{4}\frac{(\Delta x)^2}{\tau}\right)$$

$$I^5 = \frac{1}{4}\frac{h^{+2}}{\tau} - \frac{h^{+2}}{h^+ + h^-}\left(\frac{1}{h^+}\ln\frac{u_+}{u} + \frac{1}{h^-}\ln\frac{u_-}{u}\right)$$

$$I^6 = \frac{1}{4}\frac{\hat{h}^{+2}}{\tau} + \frac{\hat{h}^{+2}}{\hat{h}^+ + \hat{h}^-}\left(\frac{1}{\hat{h}^+}\ln\frac{\hat{u}_+}{\hat{u}} + \frac{1}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right)$$

$$I^7 = \frac{\Delta x h^+}{\tau} + \frac{2h^+}{h^+ + h^-}\left(\frac{h^-}{h^+}\ln\frac{u_+}{u} - \frac{h^+}{h^-}\ln\frac{u_-}{u}\right)$$

$$I^8 = \frac{\Delta x \hat{h}^+}{\tau} + \frac{2\hat{h}^+}{\hat{h}^+ + \hat{h}^-}\left(\frac{\hat{h}^-}{\hat{h}^+}\ln\frac{\hat{u}_+}{\hat{u}} - \frac{\hat{h}^+}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right).$$

Approximating the system (54) by these invariants as was done for the semilinear heat equation, we obtain a system of difference evolution equations. As an example, we present here an invariant difference model that has explicit equations for the solution and the trajectory of x:

$$\begin{cases} \Delta x = \dfrac{2\tau}{h^+ + h^-}\left(-\dfrac{h^-}{h^+}\ln\dfrac{u_+}{u} + \dfrac{h^+}{h^-}\ln\dfrac{u_-}{u}\right) \\ \left(\dfrac{u}{\hat{u}}\right)^2\exp\left(-\dfrac{1}{2}\dfrac{(\Delta x)^2}{\tau}\right) = 1 - \dfrac{4\tau}{h^+ + h^-}\left(\dfrac{1}{h^+}\ln\dfrac{u_+}{u} + \dfrac{1}{h^-}\ln\dfrac{u_-}{u}\right). \end{cases} \tag{55}$$

5. Example of an exact solution

Let us find the solution of the difference model (55) for the heat equation that is invariant with respect to the operator

$$2\alpha X_2 + X_3, \alpha = \text{constant}. \tag{56}$$

The operator (56) has three invariants: t and the expressions $u\exp\left(-\frac{x^2}{4(t+\alpha)}\right)$ and $\left(\frac{\Delta x}{\tau} - \frac{x}{t+\alpha}\right)$. So, we will seek the invariant solution in the form:

$$\begin{cases} u(x,t) = f(t)\exp\left(-\dfrac{x^2}{4(t+\alpha)}\right) \\ \dfrac{\Delta x}{\tau} = g(t) + \dfrac{x}{t+\alpha}. \end{cases}$$

Substituting this form of the solution to the system (55), we obtain ordinary difference equations for $f(t)$ and $g(t)$:

$$\begin{cases} f(t+\tau) = \left(\dfrac{t+\alpha}{t+\tau+\alpha}\right)^{1/2} f(t) \\ g(t) = 0. \end{cases} \tag{57}$$

Solving this system, we find the solution of the difference equation

$$u(x,t) = f(0)\left(\frac{\alpha}{t+\alpha}\right)^{1/2} \exp\left(-\frac{x^2}{4(t+\alpha)}\right)$$

and the solution for the evolution of a grid:

$$x = x^0\left(\frac{t+\alpha}{\alpha}\right).$$

The obtained solution is the solution of the Cauchy problem with the invariant initial condition:

$$u(x,0) = f(0)\exp\left(-\frac{x^2}{4\alpha}\right).$$

If $\alpha = 0$, the fundamental solution of the heat equation

$$u(x,t) = C\left(\frac{1}{t}\right)^{1/2} \exp\left(-\frac{x^2}{4t}\right) \tag{58}$$

is a solution of the difference model. This solution holds on the grid:

$$\Delta x = \frac{\tau}{t}x.$$

In all cases listed above the difference mesh is arbitrary at the initial point, $t = 0$. In this case it will not be uniform on other time layers. If the grid is uniform in the x–direction at $t = 0$ ($h_+ = h_- = h$), the steps of the grid in the x–direction will be equal to each other on every time layer, but differ from steps on the previous time layer.

It is necessary to mention that the obtained difference invariant solution is the solution of the corresponding differential equation that is invariant with respect to the operator (56). As in the differential case the above reduction procedure can be applied for every subalgebra of the algebra (50), and then one obtains different moving meshes which are self-adaptive to every solution.

Thus, the above difference models inherit both groups of the differential equations and the potential to be integrated on a subgroup.

6. Numerical calculations

Here we do not discuss the questions of stability and convergence of the developed schemes. These are hard questions for nonlinear schemes, but one of the ways to check them is by computing the numerical solutions to the exact solutions of the original differential equations. Below we present the results of numerical calculation of the invariant solution (58) by means of the invariant model (55). It is necessary to note that the calculations were not carried out for equations (57) reduced on the subgroup, but for the nonstationary equations (55). Initial data correspond to the solution (58) with $t = 10$. In figure 5 we present the evolution of u from invariant initial data by the invariant scheme (55).

Let us note that the difference model (55) gives us practically the exact solution for equation (49). The difference between the solution of the model (55) and the exact solution of the equation (49) is shown in figure 6.

In figure 7 the evolution of the grid in the plane (t, x) for the calculation of the solution (58) is shown.

The same calculations for the difference equation (51) on the orthogonal grid (as that in figure 1) gives a similar picture (see figure 8).

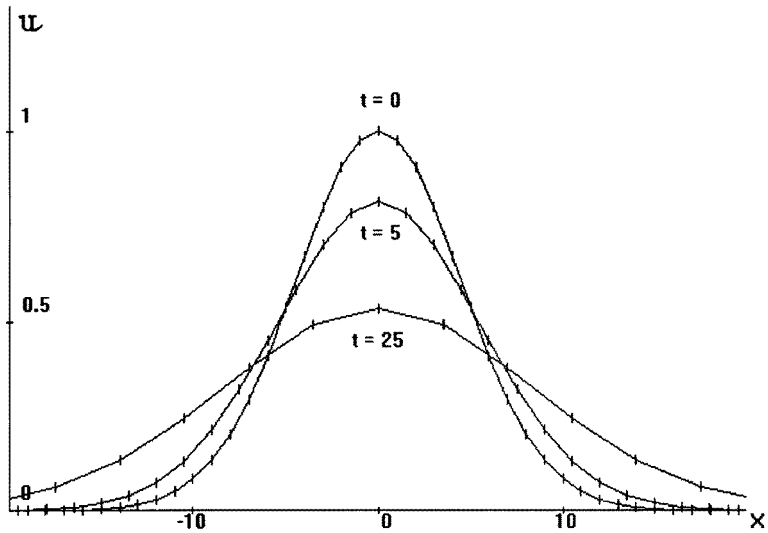

Figure 5.

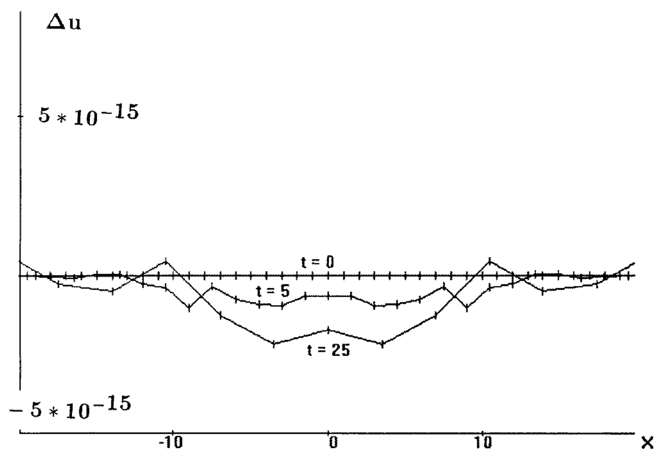

Figure 6.

In this case the solution does not coincide with exact solution of the equation (49). The difference between the exact and numerical solution is shown in figure 9.

We should note that for the calculation on the model (55) we defined the boundary values u on the moving ends of the space interval. For the difference equation (51) we defined the boundary values of u on the ends of the fixed orthogonal grid in accordance with the same solution. The comparison of the two different models shows that for the invariant model even on the decreasing number of the points of the grid on the initial space interval we have greater accuracy than for the noninvariant one.

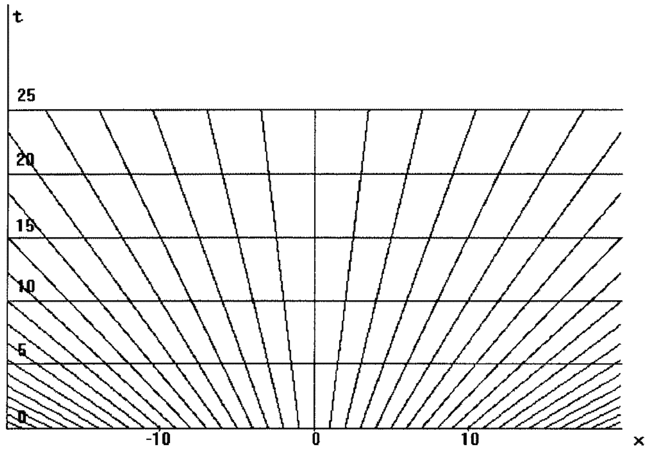

Figure 7.

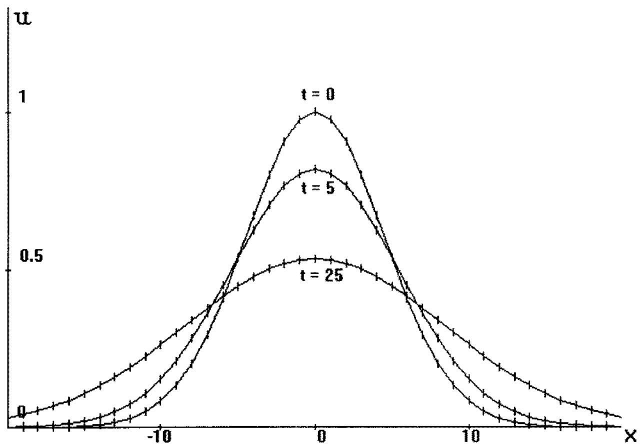

Figure 8.

7. Remarks

Following the above technique for the Burgers equation for the potential

$$w_t + \frac{1}{2} w_x^2 = w_{xx}$$

we obtain the finite-difference model for this equation on a moving mesh

$$\begin{cases} \Delta x = \tau \dfrac{h^- \underset{h}{w}_x + h^+ \underset{h}{w}_{\bar{x}}}{h^+ + h^-} \\ \exp\left(\hat{w} - w - \dfrac{\Delta x^2}{2\tau}\right) = 1 + \tau \underset{h}{w}_{x\bar{x}} \end{cases}$$

where $\underset{h}{w}_{x\bar{x}} = \frac{2}{h^- + h^+}(\underset{h}{w}_x - \underset{h}{w}_{\bar{x}})$.

It is interesting to note that this model is connected with model (55) by the same Hopf transformation

$$w = -2 \ln u$$

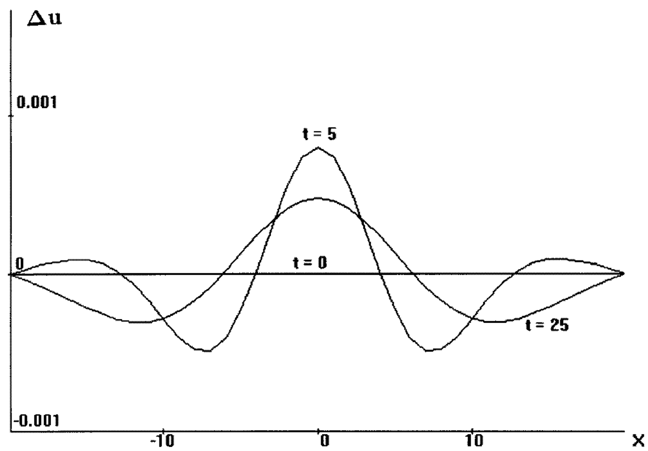

Figure 9.

as their continuous counterparts.

It is important to note that in all cases the moving in (x, t)-plane meshes can be stopped by the new coordinates of Lagrange's type with one additional dependent variable (for involvement of those coordinates see, for example, [14]).

Acknowledgments

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